

On complex H-type Lie algebras

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Abstract

H-type Lie algebras were introduced by Kaplan as a large class of real Lie algebras which, when equipped with a suitable inner product, have certain rigidity properties that naturally generalize the familiar Heisenberg Lie algebra \mathfrak{h}^3 . Among them are the complex Heisenberg Lie algebras $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ equipped with the (Hermitian) Euclidean inner product. One might ask whether there are other finite-dimensional complex Lie algebras which, when equipped with some Hermitian inner product, are H-type. In this short note, we show that there are not.

Since their introduction by Kaplan [7], H-type Lie algebras, and their corresponding nilpotent Lie groups, have attracted interest as a natural generalization of the classical real Heisenberg Lie algebra \mathfrak{h}^3 of dimension 3 and the corresponding real Heisenberg group \mathbb{H}^3 . The Heisenberg group is a motivating example in many areas of mathematics, and in many cases, facts about the Heisenberg group carry over into the H-type setting. For instance, H-type groups carry a natural structure as sub-Riemannian manifolds, and the analysis of their sub-Laplacians has attracted considerable interest. As a sampling, we mention [6, 4, 8, 5, 1]. The H-type Lie algebras are a large class; in particular, it is shown in [7] that their centers may have any finite dimension.

One notable example of an H-type Lie algebra is the complex Heisenberg Lie algebra $\mathfrak{h}_{\mathbb{C}}^3$ with its usual Euclidean inner product, which has the additional property of being a complex Lie algebra. As such, analysis on the complex Heisenberg group can take advantage of all the tools of complex geometry, together with the many results for H-type groups mentioned

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above. The same is true for the higher-dimensional complex Heisenberg (or Heisenberg–Weyl) Lie algebras $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ defined below. However, the purpose of this note is to record the elementary fact, which does not seem to have appeared previously, that there are no other examples of complex Lie algebras which are H-type under a Hermitian inner product.

We begin by recalling the definition of an H-type Lie algebra, as formulated in [2, Definition 18.1.1]. (The original definition in [7] is equivalent but slightly less convenient for our purposes.) Let \mathfrak{g} be a real finite-dimensional Lie algebra equipped with an inner product $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$. Let \mathfrak{z} be the center of \mathfrak{g} , and let $\mathfrak{v} = \mathfrak{z}^{\perp}$. For $z \in \mathfrak{z}$ and $u \in \mathfrak{v}$, define $J_z u$ as the unique element of \mathfrak{v} satisfying

$$\langle J_z u, v \rangle = \langle z, [u, v] \rangle \quad \text{for all } v \in \mathfrak{v}. \quad (1)$$

It is clear that $J_z : \mathfrak{v} \rightarrow \mathfrak{v}$ is a linear map, and moreover is linear in z .

Definition. We say that $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is **H-type** if the following two conditions hold:

1. $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$
2. For each $z \in \mathfrak{z}$ with $\|z\| = 1$, $J_z : \mathfrak{v} \rightarrow \mathfrak{v}$ is an isometry with respect to $\langle \cdot, \cdot \rangle$.

We observe that an H-type Lie algebra is necessarily nilpotent of step 2.

Now suppose that \mathfrak{g} is a complex Lie algebra, whose complex structure we denote by i . If we wish to equip \mathfrak{g} with a real inner product, it is natural to demand some compatibility with the complex structure. Specifically, we would like the inner product to be **Hermitian**, i.e., for $x, y \in \mathfrak{g}$ we have $\langle ix, iy \rangle = \langle x, y \rangle$. We may then define J as in (1). We observe for later use that, as a consequence of the Hermitian property of the inner product, we have for $\alpha, \beta \in \mathbb{C}$ and $u, z \in \mathfrak{g}$,

$$J_{\alpha z}(\beta u) = \alpha \bar{\beta} J_z u. \quad (2)$$

That is, $J_z u$ is complex linear in z and conjugate linear in u .

We may then speak of a **complex H-type Lie algebra** as a complex Lie algebra \mathfrak{g} equipped with a Hermitian inner product $\langle \cdot, \cdot \rangle$ such that $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is H-type in the sense of the above definition.

Example. The **complex Heisenberg Lie algebra** of complex dimension $2n + 1$ is the complex Lie algebra $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ generated (over \mathbb{C}) by vectors $\{x_1, y_1, \dots, x_n, y_n, z\}$ with the bracket defined by $[x_k, y_k] = z$, and for $j \neq k$,

$[x_j, y_k] = [x_j, z] = [y_j, z] = 0$. We may equip $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ with the real inner product $\langle \cdot, \cdot \rangle$ that makes all of $x_k, ix_k, y_k, iy_k, z, iz$ orthonormal; it is clear that this inner product is Hermitian. The center \mathfrak{z} of $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ is spanned (over \mathbb{C}) by z , so we clearly have $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$. Defining J as above, it is easy to compute

$$J_z x_k = y_k \quad J_z y_k = -x_k \quad J_z ix_k = -iy_k \quad J_z iy_k = ix_k$$

so that J_z is an isometry. Moreover, every element $w \in \mathfrak{z}$ is of the form $w = \alpha z$ for some $\alpha \in \mathbb{C}$, and $\|w\| = |\alpha|$, so using (2) we see that J_w is an isometry whenever $\|w\| = 1$. Thus $(\mathfrak{h}_{\mathbb{C}}^{2n+1}, \langle \cdot, \cdot \rangle)$ is a complex H-type Lie algebra.

We shall now prove that these are, up to isometric isomorphism, the only complex H-type Lie algebras.

We first recall the well-known Clifford algebra identity for H-type Lie algebras:

$$J_z J_w + J_w J_z = -2 \langle z, w \rangle I, \quad z, w \in \mathfrak{z}. \quad (3)$$

To prove this, first consider the case when $w = z$ and $\|z\| = 1$. Then for any $u, v \in \mathfrak{v}$, we have

$$\langle J_z^2 u, v \rangle = \langle z, [J_z u, v] \rangle = -\langle z, [v, J_z u] \rangle = -\langle J_z v, J_z u \rangle = -\langle v, u \rangle$$

since J_z is an isometry. So $J_z^2 = -I$. The general case follows by scaling and polarization.

Theorem 1. *Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ be a complex H-type Lie algebra as defined above. Then for some n , $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is isometrically isomorphic to $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ with its standard Hermitian inner product.*

In particular, complex H-type Lie algebras are completely classified by their dimension, and all have centers of complex dimension 1. Of course, the analogous statement holds for their corresponding nilpotent complex Lie groups, for which the inner product corresponds to a left-invariant Riemannian metric which is Hermitian.

Proof. Suppose $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is complex H-type, and let \mathfrak{v} , \mathfrak{z} and J be defined as above.

We first note that \mathfrak{z} must have complex dimension 1. If not, then we can find $z, w \in \mathfrak{z}$ with $\|z\| = \|w\| = 1$ and $\langle z, w \rangle = \langle iz, w \rangle = 0$. Then by (3) and (2) we have

$$\begin{aligned} 0 &= -2 \langle z, w \rangle I = J_z J_w + J_w J_z \\ 0 &= -2 \langle iz, w \rangle I = J_{iz} J_w + J_w J_{iz} = i J_z J_w + J_w i J_z = i(J_z J_w - J_w J_z). \end{aligned}$$

Thus $J_w J_z = J_z J_w = 0$, contradicting the requirement that J_z, J_w be isometries.

Therefore, \mathfrak{z} is the complex span of a single unit vector z . We recursively construct an orthonormal basis for \mathfrak{v} of the form $\{x_k, ix_k, y_k, iy_k : k = 1, \dots, n\}$. Suppose $\{x_k, ix_k, y_k, iy_k : k = 1, \dots, m-1\}$ have been constructed and do not span \mathfrak{v} . Let x_m be any unit vector orthogonal to all of x_k, ix_k, y_k, iy_k for $k = 1, \dots, m$. Then set $y_m = J_z x_m$. We have $\|y_m\| = 1$, and a few straightforward computations verify that $\{x_k, ix_k, y_k, iy_k : k = 1, \dots, m\}$ are now orthogonal. When the process terminates, we have our orthonormal basis.

To compute brackets, for $j \neq k$ we have

$$\begin{aligned}\langle z, [x_k, y_k] \rangle &= \langle J_z x_k, y_k \rangle = \langle y_k, y_k \rangle = 1 \\ \langle z, [x_k, x_j] \rangle &= \langle J_z x_k, x_j \rangle = \langle y_k, x_j \rangle = 0 \\ \langle z, [y_k, y_j] \rangle &= \langle J_z y_k, y_j \rangle = \langle J_z y_k, J_z x_j \rangle = \langle y_k, x_j \rangle = 0 \\ \langle z, [x_k, y_j] \rangle &= \langle J_z x_k, y_j \rangle = \langle y_k, y_j \rangle = 0.\end{aligned}$$

Similar computations show that if z is replaced by iz , all of the above expressions vanish. Each bracket is in \mathfrak{z} and hence a complex scalar multiple of z , so we have

$$[x_k, y_k] = z, \quad [x_k, x_j] = [y_k, y_j] = [x_k, y_j] = 0.$$

The corresponding brackets for ix_k, iy_k , etc, follow from the complex bilinearity of the bracket. These are precisely the same relations as for the complex Heisenberg Lie algebra $\mathfrak{h}_{\mathbb{C}}^{2n+1}$, and the basis is orthonormal, just as for the standard inner product on $\mathfrak{h}_{\mathbb{C}}^{2n+1}$. Therefore, $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is isometrically isomorphic to $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ with its standard inner product. \square

Remark. It is well known that, given an H-type Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, the map $\mathfrak{z} \ni z \mapsto J_z \in \text{End}(\mathfrak{v})$ extends to a representation on \mathfrak{v} of the Clifford algebra $Cl(\mathfrak{z}, \langle \cdot, \cdot \rangle)$. This gives a one-to-one correspondence between H-type Lie algebras and Clifford algebra representations. (For further details, see [3, Section 2.2].) As such, Theorem 1 could be restated in terms of a real Clifford algebra \mathcal{A} with a representation on a real vector space V , each of which is equipped with additional structure corresponding to the complex structure on \mathfrak{g} . However, this does not appear to be terribly enlightening. In particular, we cannot simply state Theorem 1 in terms of a complex Clifford algebra with a complex representation, because as noted in (2), the map J_z is not complex linear on \mathfrak{v} but instead conjugate linear, so it doesn't correspond naturally to an endomorphism of the complex vector space \mathfrak{v} .

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